THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. AUGUST-SEPTEMBER, 1898. No. 8-9.

BIOGRAPHY.

RENÉ DESCARTES.

BY B. F. FINKEL.

RENÉ DESCARTES, the first of the modern school of mathematicians, was born at La Haye, a small town on the right bank of the Creuse and about midway between Tours and Poitiers, on March 31st, 1596, and died at Stockholm, on February 11th, 1650. "The house is still shown where he was born, and a *métairie* about three miles off still retains the name of Les Cartes. His family on both sides was of Poitevin descent and had its headquarters in the neighboring town of Châtterault, where his grandfather had been a physician. His father, Joachim Descartes, purchased a commission as counsellor in the Parlement Rennes and thus introduced the family into that demi-noblesse of the robe of which, in stately isolation between the bourgeoisie and the high nobility, maintained a lofty rank in the hierarchy of France. For one-half of each year required for residence the elder Descartes removed, with his wife, Jeanne Brochard, to Rennes. Three children, all of whom first saw the light at La Haye, sprang from the union,—a son, who afterwards succeeded to his father in the Parlement, a daughter who married a M. du Crevis, and a second son, René. His mother, who had been ailing beforehand, never recovered from her third confinement; and the motherless infant was intrusted to a nurse, whose care Descartes in after years remembered by a small pension."*

Britannica Encyclopedia, Ninth Edition.

Descartes, who early showed an inquisitive mind, was called by his father, "my philosopher." At the age of eight, Descartes was sent to the school of La Flèche, which Henry IV had lately founded and endowed for the Jesuits, and here he continued from 1604 to 1612. Of the education here given, of the equality maintained among the pupils, and of their free intercourse, he spoke at a later period in terms of high praise. Descartes himself enjoyed exceptional privileges. His feeble health excused him from the morning duties, and thus early he acquired the habit of matutinal reflection in bed, which clung to him throughout life. When he visited Pascal in 1647, he told him that the only way to do good work in mathematics and to preserve his health was never to allow any one to make him get up in the morning before he felt inclined to do so. Even at this period he had begun to distrust the authority of tradition and his teachers.

Two years before leaving school (1610) he was selected as one of twenty-four gentlemen who went forth to receive the heart of the murdered king as it was borne to its resting place at La Flèche. During the winter of 1612, he completed his preparations for the world by lessons in horsemanship and fencing; and then in the spring of 1613 he started for Paris to be introduced to the world of fashion. Fortunately the spirit of dissipation did not carry him very far, the worst being a passion for gaming. Here through the medium of the Jesuits he made the acquaintance of Mydorge, one of the foremost mathematicians of France, and renewed his schoolboy friendship with Father Mersenne, and together with them he devoted the two years of 1615 and 1616 to the study of mathematics.

"The withdrawal of Mersenne in 1614 to a post in the provinces was the signal for Descartes to abandon social life and shut himself up for nearly two years in a secluded house of the Faubourg St. Germain. Accident, however, betrayed the secret of his retirement; he was compelled to leave his mathematical investigations and to take a part in entertainments, where the only thing that chimed in with his theorizing reveries was the music. The scenes of horror and intrigue which marked the struggle for supremacy between the various leaders who aimed at guiding the politics of France made France no fit place for a student and held out little honorable prospect for a soldier. Accordingly, in May, 1617, Descartes, now twenty-one years of age, set out for the Netherlands, and took service in the army of Prince Maurice of Orange, one of the greatest generals of the age, who had been engaged for some time in a war with the Spanish forces in Belgium. At Breda, he enlisted as a volunteer, and the first and only pay which he accepted he kept as a curiosity through life. There was a lull in the war; and the Netherlands were distracted by the quarrels of Gomarists and Arminians. During the leisure thus arising, Descartes one day, as he roved through Breda, had his attention drawn to a placard in the Dutch tongue; and as the language of which he never became perfectly master, was then strange to him, he asked a bystander to interpret it in either French or Latin. The stranger, who happened to be Isaac Beeckman, principal of the College of Dort, offered with some surprise to do so into Latin, if the inquirer would bring him a so-

lution of the problem,—for the advertisement was one of those challenges which the mathematicians of the age, in the spirit of the tournament of chivalry, were accustomed to throw down to all comers, daring them to discover a geometrical mystery known as they fancied to themselves alone. Descartes promised and fulfilled; and a friendship grew up between him and Beeckman—broken only by the literary dishonesty of the latter, who in later years took credit for the novelty contained in a small essay on music (Compendium Musicae) which Descartes wrote at this period and intrusted to Beeckman."*

The unexpected test of his mathematical attainments afforded by the solution of the problem referred to, its solution costing him only a few hours study, made the uncongenial army life distasteful to him, but under family influence and tradition, he remained a soldier, and was pursuaded at the commencement of the thirty years' war to volunteer under Count de Bucquoy in the army of Bavaria. The winter of 1619, spent in quarters at Neuburg on the Danube, was the critical period in his life. Here, in his warm room (*dans un poele*), he indulged those meditations which afterwards led to the *Discours de la Méthode* (Discourse of Method). It was here that, on the eve of St. Martin's day, November 10, 1619, he "was filled with enthusiasm, and discovered the foundations of a marvelous science."

He retired to rest with anxious thoughts of his future career, which haunted him through the night in three dreams, that left deep impressions on his mind. "Next day," he says, "I began to understand the first principles of my marvelous discovery." Thus the date of his philosophical conversion is fixed to a day. This day marks the birth of modern mathematics. His discovery, viz., the coöperation of ancient geometry and algebra, is epoch-making in the history of mathematics.

It is frequently stated that Descartes was the first to apply algebra to geometry. This statement is not true, for Vieta and others had done this before him, and even the Arabs sometimes used algebra in connection with geometry. "The new step that Descartes did take was the introduction into geometry of an analytical method based on the notion of variables and consonants, which enabled him to represent curves by algebraic equations. In the Greek geometry, the idea of motion was wanting, but with Descartes it became a very fruitful conception. By him a point was determined in position by its distances from two fixed lines or axes. These distances varied with every change of position in the point. This geometric idea of *co-ordinate representation* together with the algebraic idea of *two variables in one equation* having an indefinite number of simultaneous values, furnished a method for the study of loci, which is admirable for the generality of its solutions. Thus the entire conic sections of Appollonius is wrapped up and contained in a single equation of the second degree."†

"Descartes found in mathematics, as did Kant and Comte, the type of all faultless thought; and he proved his appreciation of his insight by the invention

Encyclopedia Britannica, Ninth Edition.
†*Cajori's History of Mathematics.*

of a new symbolic mechanism and artifice for the applications of algebra to geometry (*Analytic Geometry*, as it is now called, which, in a growing sense, let it be said, existed before him), and by his discoveries in the theory of equations, which were fundamental in their importance."*

After a short sojourn in Paris, Descartes moved to Holland, then at the height of its power. There for twenty years he lived, giving up all his time to philosophy and mathematics. Science, he says, may be compared to a tree; metaphysics is the root, physics is the trunk, and the three chief branches are mechanics, medicine, and morals, these forming the three applications of our knowledge, namely, to the external world, to the human body, and to the conduct of life; and with these subjects alone his writings are concerned.

He spent the time from 1629 to 1633 writing *Le Monde*, a work embodying an attempt to give a physical theory of the universe; but finding its publication likely to bring on him the hostility of the Church, and having no desire to pose as a martyr, he abandoned it. The incomplete manuscript was published in 1664.

He then devoted himself to composing a treatise on universal science; this was published at Leyden in 1637 under the title Discourse de la méthode pour bien conduire sa raison et chercher la verité dans les sciences, and was accompanied with three appendices entitled La Dioptrique, Les Méléores, and La Géométrie. It is from the last of these that the invention of analytical geometry dates. In 1641, he published a work called Meditations, in which he explained at some length his views of philosophy as sketched out in the *Discourse*. In 1644, he issued the Principia Philosophiae, the greater part of which was devoted to physical science especially the laws of motion and the theory of vortices. In his theory of vortices, he commences with a discussion of motion; and then lays down ten laws of nature, of which the first two are almost identical with the first two as laid down by Newton. The remaining eight are inaccurate. He next proceeds to a discussion of the nature of matter which he regards uniform in kind though there are three forms of it. He assumes that the matter of the universe is in motion, that this motion is constant in amount, and that the motion results in a number of vortices. He states that the sun is the center of an immense whirlpool of this matter, in which the planets float and are swept round like straws in a whirlpool of water.

Each planet is supposed to be the center of a secondary whirlpool by which its satellites are carried, and so on. All of these assumptions are arbitrary and unsupported by any investigation. It is a little strange that a man who began his philosophical reasonings by doubting all things and finally coming to *cogito, ergo sum* should have made assumptions so groundless.

While Descartes was a philosopher of a very high type, yet his fame will ever rest on his researches in mathematics. The first important problem solved by Decartes in his geometry is the problem of Pappus, viz.: "Given several straight lines in a plane, to find the locus of a point such that perpendiculars, or,

*The Open Court, August, 1898.

more generally, straight lines at given angles, drawn from the point to the given lines, shall satisfy that the product of certain of them shall be in given ratio to the product of the rest." "The most important case of this problem is to find the locus of a point such that the product of the perpendiculars on m given lines be in a constant ratio to the product of the perpendiculars on n other given straight lines. The ancients had solved this geometrically for the case $m=1$, $n=1$, and the case $m=1$, $n=2$. Pappus had further stated that if $m=n=2$, the locus was a conic, but he gave no proof; Descartes also failed to prove this by pure geometry, but he showed that the curve was represented by an equation of the second degree, that is, was a conic; subsequently Newton gave an elegant solution of the problem by pure geometry."*

In algebra, Descartes expounded and illustrated the general methods of solving equations up to those of the fourth degree (and believed that his method could go beyond), stated the law which connects the positive and negative roots of an equation with the change of signs in the consecutive terms, known as Descartes' Law of Signs, and introduced the method of indeterminate coefficients for the solution of equations.

In appearance, Descartes was a small man with large head, projecting brow, prominent nose, and black hair coming down to his eyebrows. His voice was feeble. Considering the range of his studies he was by no means widely read, had no use for Greek, as is shown by his disgust when he found that Queen Christina devoted some time each day to its study, and despised both learning and art unless something tangible could be extracted therefrom. In philosophy, he did not read much of the writings of others. In disposition, he was cold and selfish, He never married, and left no descendants, though he had one illegitimate daughter, Francine, who died in 1640, at the age of five.

In 1649, through the instigation of his close personal friend, Chanut, he received an invitation to the Swedish court, and in September of that year he left Egmond for the north. Here, on the 11th of February, 1650, he died of inflamation of the lungs brought about by too close devotion to the sick-room of his friend Chanut, who was dangerously ill with the same disease.

---

*Ball's *Short Account of the History of Mathematics.*

ON SEVERAL POINTS IN THE THEORY OF THE GROUPS OF A FINITE ORDER.*

By DR. G. A. MILLER.

1. *Definition.* The most prominent condition which the operators of a group must satisfy is that the product of any two of them is equivalent to a single operator contained in the group. When the operators of the group are represented by substitutions it is not necessary to add any additional condition, *i. e.* a set of substitutions which includes every substitution that is obtained by multiplying any two of the set together or by squaring any one of them forms a group, provided that no two of the substitutions are identical.

It is however not customary to call *any* set of operators that satisfies the given condition a group : *e. g.* the three numbers 0, 1, -1 clearly satisfy this condition while they would not generally be said to form a group when they are combined by multiplication. Some writers seem to make the definition of group so general as to include this case,† but the theory of the groups of operators which satisfy this general definition without also satisfying the more restricted ones remains to be developed.

In addition to the given conditions the operators of a group are usually required to satisfy the associative law, and the laws that the product of any two of them is completely determined by the operators and the method of combining them and that from the equations $ab=ac$, $\alpha\beta=\gamma\beta$ it must follow respectively that $b=c$, $\alpha=\gamma$; where a, b, c, α, β, γ represent any operators of the group.

While the method of combining the operators of a group is generally called multiplication yet it should not be inferred that any restrictions are imposed upon this method except those given above, so that the term multiplication in this connection merely implies some definite law of combination : *e. g.* if we combine the following n numbers, 0, 1, 2, 3, ... , $n-1$, by adding any one to each of them, their positive remainders with respect to modulus n will determine the cyclical group of order n and every cyclical group can be represented in this way. If we combine these numbers by multiplication, taking their positive remainders according to the same modulus, they do *not* form a group in the usual sense of this term. It is thus clear that the group property is not inherent in a set of operators but, that it exists, in part, in the law of combination.

As may be inferred from the heading, these remarks relate to the groups of a finite order. This restriction seemed appropriate for this occasion since my investigations have been confined to this class of groups. In recent years the groups of an infinite order have received a great deal of attention. Among the investigators of the discontinuous groups of an infinite order Poincaré, Klein and Fricke are the most prominent while Sophus Lie is preëminent among the many investigators of continuous groups. After this glance at these two sturdy

*Read before the *Oliver Mathematical Club* of Cornell University as initiatory paper, February, 1898.
†Cf. Klein's Ikosaeder, page 5; Miss Scott's Modern Analytical Geometry, page 261; etc.

offsprings of our subject we proceed to give a short sketch of its development.

Historical Sketch. While the group concept appears early in the development of mathematics yet it was not until substitution groups began to be studied that this concept began to receive considerable attention and that its fundamental importance began to be recognized. Writers do not agree in regard to the founder of the theory of substitutions. Some ascribe this honor to Lagrange, others to Abel, others to Galois, and still others to Cauchy. In recent years a considerable number, under the leadership of Burkhardt,* have been contending that the origin of the theory of substitutions is found in an Italian work bearing the title "*Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto*, di Paolo Ruffini," Bologna, 1799.

This work contains some fundamental concepts such as transitivity, intransitivity, primitivity, etc., that are generally attributed to the later works of Cauchy. Its proof that the general equation whose degree exceeds four is not algebraically solvable lacks rigor. The first vigorous proof of this important theorem, which engaged the attention of mathematicians for centuries, was given by Abel and it was published in the first volume of Crelle, the oldest German mathematical paper still living. As this proof was partly based upon the theory of substitutions it drew considerable attention to this subject. This seems to be the main reason why some writers, Hagen for example, call Abel one of the founders of the theory of substitutions.†

The next great impulse to the study of this theory was given by that very remarkable young mathematician Galois, who proved that every algebraic equation belongs to a group and that its solvability by the extraction of roots depends upon the factors of composition of this group. This discovery united the theory of equations very closely with the theory of substitution groups. The French mathematician Jordan has been especially prominent in pointing out results which depend upon this relation. The subject is one of great difficulty and it will probably furnish a fertile field of investigation for many years to come.

The claims that Cauchy was the founder of the theory of substitutions seem to be based upon the fact that he was the first to write extensively on this subject apart from any direct applications. His *Exercices d' Analyse*, vol. 3, (1844), contains the first systematic treatment of this theory and contributed very largely towards making it known to a larger class of students. Although some of the concepts which have generally been attributed to him have recently been traced to earlier writings yet there remains much which is undoubtedly due to Cauchy and some of these facts are of fundamental importance.

In recent years there has been developing an offspring from substitution groups, which resembles the parent more closely than the two mentioned above. I refer to the operation or abstract groups. In the first volume of the *American Journal of Mathematics* Cayley calls attention to the fact that a group is an ab-

*Schlomilch's *Zeitschrift*, 1892, supplement, page 159.

†*Synopsis der Mathematik*, page 281.

stract concept and that it should be defined "by means of the laws of combination of its symbols." A few years later Dyck published several articles in the *Mathematische Annalen* in which he laid the foundation of a part of the theory of abstract groups.

As many of the properties of these groups can be readily studied by means of substitution groups their development has given a new impulse to the study of substitution groups. This is the modern impulse and it is to be hoped that it will lead to great activity in this line. The bulk of the recent publications on groups of a finite order have been along the lines of abstract groups and these groups promise to remain a fertile field of investigation for a number of years at least.

Standing Problems. While knowledge is cosmopolitan and scientific discoveries belong to the world rather than to a particular nation yet it does not behoove a great nation or even a great university to be satisfied to borrow all its scientific facts from the rest of the world. Such a state of affairs indicates stagnation and is evidence either of inability or of want of patriotism. It also deprives the students of the inspiration and joy which attend the discovery of important scientific facts that tend to contribute something towards enriching all future generations.

It is evident that the standing problems of a comparatively new subject can be reached more readily and are generally less difficult than those of older subjects. Hence the newer subjects are the more inviting to the young investigator. While our modern mathematical journals do a great deal towards aiding the student to find desirable unsolved problems yet some of the simpler ones are solved at such a rapid rate that it is scarcely possible to keep in touch with a large portion of the region in which these discoveries are being made.

It has been observed for a long time that the problem to find all the substitution groups of any degree would be of the greatest importance to algebra.* This problem is far from a complete solution. It has been solved for all the degrees less than eleven but the methods employed are not suitable for very large degrees and they throw very little light on the general problem. The general theorems on this subject are still few, and the progress that has been made in recent years under the inspiration of the great prize of the Paris Institute gives evidence of the great difficulty of the problem rather than any hope of its early solution.

In recent years all the simple groups of the finite continuous groups have been found. It would be a great step forward if all the simple groups of the groups of a finite order could also be found. Several new infinite systems of such groups have recently been discovered by Professors Moore and Burnside, and by Dr. Dickson but the progress towards the complete solution of this problem is exceedingly slow. Any discoveries that throw light on this subject are of great interest.

As problems of somewhat smaller interest in themselves we may mention

*Cf. Serret's *algebra superieure*, 3rd edition, page 256.

the proof of the existence or non-existence of simple groups of an odd order or of order $p^\alpha q^\beta$, p and q being prime numbers; the superior limit of transitivity of primitive groups that do not contain the alternating group; the simplification of the methods of proving the solvability or the insolvability of a group, etc.

REPLY TO PROFESSOR FISK'S CRITICISM OF A CERTAIN FEATURE OF NICHOLSON'S CALCULUS.

By J. W. NICHOLSON, A. M., LL. D., Professor of Mathematics, Louisiana State University, Baton Rouge, La.

In the March number of the *Bulletin* is a brief review of my Calculus, by Professor Fiske, of which the following is an extract:

"In another note (A_3) at the end of the work the author critizes the grounds assigned by Byerly and by Rice and Johnston for making $d(dx)=0$. He contends that the differential of dx is zero, because dx as a variable is independent of x. This, of course, is not sound. If a variable y is independent of another variable x, it is true that we may still write

$$dy=\frac{dy}{dx}dx;$$

but the coefficient of dx is not a partial derivative, and dy, therefore, instead of being zero is indeterminate. In order that $d(dx)$ may be zero, we must assume that dx takes the same value for all values of x. This assumption, however, does not prevent our varying dx from one instant to another in a perfectly arbitrary manner."

As the question involved is an interesting and important one, and believing that Professor Fiske had not fairly presented my discussion of the point at issue, I wrote a brief reply to the above criticism, and sent it to the *Bulletin* for publication. Several weeks thereafter my reply was returned to me without publication and with the following additional stricture:

"The author fails to realize that if $dy=0$ when x goes from x to $x+dx$ then y is not completely independent of x, but has such a dependence that it does not alter when x alters."

The question involved is not whether dx is a quantity whose total differential is 0, but whether it is a quantity whose differential *with respect to* x is 0.

Of course if x and y are two variables which are independent of each other, and dx be the total differential of the one and dy that of the other, and $\frac{dy}{dx}$ is understood to mean the ratio of these differentials, $\frac{dy}{dx}$ is not 0 but indeterminate, as Professor Fiske says. Or again, under the same hypothesis, "if $dy=0$

when x goes from x to $x+dx$'' y would depend on x in the manner indicated by the last criticism.

But the point in question comes up in proving that $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, *i. e.* the differential coefficient of $\frac{dy}{dx}$ with respect to x, $=\frac{d^2y}{dx^2}$; and in this demonstration we have no occasion to consider whether dx is a quantity whose total differential is 0 or not. It is a differentiation with respect to x which is indicated by $\frac{d}{dx}$; and it is to be demonstrated that if this operation be performed on $\frac{dy}{dx}$, on the hypothesis that this symbol may be treated as a fraction whose terms are dy and dx, and the understanding that d^2y *represents the differential of* dy *with respect to* x, the result is $\frac{d^2y}{dx^2}$. Thus:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dx\,\frac{d}{dx}(dy)-dy\,\frac{d}{dx}(dx)}{dx^2}=\frac{d^2y}{dx^2},$$

because $d^2y=\frac{d}{dx}(dy).dx$, by definition, and $\frac{d}{dx}(dx)=0$, *for the same reason that the differential coefficient with respect to* x *of any variable which is independent of* x *is* 0. $\frac{d}{dx}(dx)$ does not mean the ratio of any variation that dx may be supposed to undergo, to dx, but the ratio of the variation which dx undergoes *in consequence of a variation in* x, to dx. To say that $\frac{d}{dx}(dx)=0$ is not therefore to say that dx is a constant, but merely that it undergoes no variation in consequence of a variation in x. Indeed, dx may have any value of x, and is therefore a variable independent of x, and being *independent*, it may be regarded and treated as an *absolute constant* except in cases where the independence of x would thereby be destroyed, as shown in my Calculus.

Precisely the same considerations are involved in the derivation of the equation $d^2y=f''(x)dx^2$ from $dy=f'(x)dx$.

The reply to Professor Fiske is therefore that in his equation, $dy=\frac{dy}{dx}dx$, for the case before us, viz: $d^2x=\frac{d}{dx}(dx).dx$, the coefficient of dx *is a partial derivative.*

The reply to the last criticism of the *Bulletin* is that in the case before us the dy is not any variation that y may be supposed to undergo while x varies from x to $x+dx$, but the variation which y undergoes in consequence of this variation in x, and this of course must be 0 if y is independent of x, whatever may be the value of y as x goes from x to $x+dx$.

The criticism in the *Bulletin* is therefore based upon a misconception of

the author's meaning, and is due to an apparent failure on the part of Professor Fiske to realize that the question is not what must be in order that $d(dx)$ may be 0, but what is in order that $\frac{d}{dx}(dx)$ may be 0.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

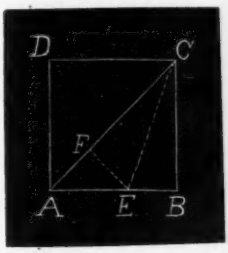
ARITHMETIC.

96. Proposed by RAYMOND SMITH, Tiffin, Ohio.

How many acres in a square field whose diagonal is 10 rods longer than the side?

I. Solution by J. F. TRAVIS, Student at Ohio State University, Columbus, O.; EDWARD R. ROBBINS, Master of Mathematics, Lawrenceville School, Lawrenceville, N. J.; J. SCHEFFER, A. M., Hagerstown, Md.; F. R. HONEY, Ph. B., New Haven, Conn.; M. E. GRABER, Mt. Eaton, O.; WALTER HUGH DRANE, Professor of Mathematics, Jefferson College, Washington, Miss.; and JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Let $ABCD$ be the square field, and AC its diagonal. On AC lay off CF equal to BC. At F erect FE perpendicular to AC and intersecting AB in E. Draw EC. Then in the right triangles CFE and CBE, CB equals CF, by construction and CE is common. Hence, FE equals EB. In the right triangle AFE, the angle FAE is equal to 45°. Hence, the angle FEA equals 45°. Hence the side AF equals the side FE. Then



$$AB=(AE+EB)=[\sqrt{(AF^2+FE^2)}+EB]$$

$$=[\sqrt{(2EF^2)}+EB]=(EF\sqrt{2}+EB)=(\sqrt{2}+1)EB.$$

But $EB=AF=10$ chains. $\therefore AB=10(\sqrt{2}+1)$, and area of the field $=AB^2=100(\sqrt{2}+1)^2=100(3+2\sqrt{2})=582.8427$ square rods, or 3.642 acres.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics, Chester High School, Chester, Pa.; and P. S. BERG, Superintendent of Schools, Larimore, N. D.

We will solve generally by making a=the excess of the diagonal over the side.

Let x=side of square field. Then $2x^2=(x+a)^2$.

Solving this equation for x, gives, $x=a(1\pm\sqrt{2})$.

$\therefore$ area $=x^2=a^2(3\pm2\sqrt{2})$.

Now substituting 10 for a, and we obtain

Area $=100(3\pm2\sqrt{2})=582.8427+$ square rods, or $17.157+$ square rods,

$=3.6429+$ acres, or $.10723+$ acres.

[QUERY. How interpret the second result? M. A. G.]

[NOTE.—The answer to this query is simply this: The second result is not geometrically interpretable, for the reason that the negative value of x from which it is derived, is not geometrically intrepretable. The negative value of x satisfies the algebraic condition expressed by the equation and that alone. ED. F.]

III. Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.; COOPER D. SCHMIDT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and CHARLES C. CROSS, Libertytown, Md.

The diagonal of a square = the side $\times \sqrt{2}$.

$\therefore$ side $\times \sqrt{2}$ = side + 10 rods, or side$(\sqrt{2}-1)$ = 10 rods, and side $= 10 \div (\sqrt{2}-1)$.

Hence $[10/(\sqrt{2}-1)]^2 \div 160 = 3.64+$, the number of acres.

97. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

In what time will \$4000 amount to \$5134.96, interest at 6% payable annually?

Solution by WILLIAM W. CHAMPLAIN, Wickford, R. I.

The time is evidently between 4 and 5 years.

The interest on \$4000 for 4 years, at 6% is . .	\$960.00
The interest on \$240 for (3+2+1) years = 6 years is .	86.40
	\$1046.40

\$1134.96 − \$1046.40 = \$88.56, the interest on \$4000 for the number of months and days exceeding 4 years, plus the interest on four unpaid installments of annual interest for the same period; that is, on \$4000 + \$960 or \$4960. Interest on \$4960 at 6% for one year is \$297.60; to gain \$88.56 it would require $\frac{8856}{29760}$ of a year, or 3 months, 17.13 days.

$\therefore$ \$4000 would amount to \$5134.96, at 6% annual interest in 4 years, 3 months, 17.13 days.

Also solved by *G. B. M. ZERR*, and *P. S. BERG*.

98. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A poor man borrowed \$20 which he repaid in eleven monthly installments of \$2 each; what was the annual rate of interest (reckoned as simple interest)?

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

The last installment would be paid in eleven months, which, therefore, would be the time the \$20 would be on interest, and the interest would be $\frac{1}{12} \times 20 \times$ rate. The first installment would be on interest 10 months, and the interest on the installments would be $\frac{55}{12} \times 2 \times$ rate. Then $\frac{220}{12}$ times the rate less $\frac{110}{12}$ times the rate would be \$2, or \$2 is $\frac{110}{12}$ times the rate. Hence the rate is $21\frac{9}{11}$.

Also solved by *WALTER H. DRANE*.

[NOTE. P. S. Berg solved problem 95, but his solution reached us too late for credit in last issue. Problem 99, should read, "How many cats will catch 100 rats in 100 minutes?" Ed. F.]

ALGEBRA.

84. Proposed by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

On the present electoral basis, if all the electoral votes of each state are cast solid for one or the other of two presidential candidates, how many combinations of states are possible for a total of 273 votes for the winning candidate?

No solution of this problem has been received.

85. Proposed by J. M. COLAW, A. M., Monterey, Va.

Sum the infinite series,

$$\frac{1}{1^2.2^2.3^2.4^2.5^2}+\frac{1}{2^2.3^2.4^2.5^2.6^2}+\frac{1}{3^2.4^2.5^2.6^2.7^2}+, \text{ etc.}$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science in Chester High School, Chester, Pa.

Let $V_m=1/[n^2(n+1)^2.....(n+m)^2]$.

Then $V_m=\frac{2m-1}{m^3}\left[\frac{1}{n^2.(n+1)^2....(n+m-1)^2}+\frac{1}{(n+1)(n+2)^2....(n+m)^2}\right]$

$$+\frac{1}{m^3(m-1)}\left[\frac{2}{(n+1)^2....(n+m-1)^2}-\frac{1}{n^2....(n+m-2)^2}\right.$$

$$\left.-\frac{1}{(n+2)^2....(n+m)^2}\right].$$

Now let $S_m=\Sigma V_m$.

$$\therefore\ S_m=\frac{2m-1}{m^3}\left[S_{m-1}+\left(S_{m-1}-\frac{1}{m^2\,!}\right)\right]$$

$$+\frac{1}{m^3(m-1)}\left[\left(2S_{m-2}-\frac{2}{(m-1)^2\,!}\right)-S_{m-2}-\left(S_{m-2}-\frac{1}{(m-1)^2\,!}-\frac{1}{m^2\,!}\right)\right].$$

$$\therefore\ S_m=\frac{4m-2}{m^3}.S_{m-1}-\frac{2m-1}{m^3.(m^2)!}-\frac{1}{m^3(m-1)}\left[\frac{1}{(m-1)^2!}-\frac{1}{m^2\,!}\right]$$

$$=\frac{4m-2}{m^3}S_{m-1}-\frac{2m-1}{m^3.(m^2)\,!}-\frac{m^2-1}{m^3(m-1).(m^2)\,!}$$

$$=\frac{4m-2}{m^3}S_{m-1}-\frac{1}{m^3.(m^2)!}\left[(2m-1)+(m+1)\right]$$

$$=\frac{4m-2}{m^3}S_{m-1}-\frac{1}{m^2.(m^2)!}=\frac{1}{m^2}\left[\frac{4m-2}{m}S_{m-1}-\frac{3}{m^2\,!}\right.$$

Now $S_0=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\frac{\pi^2}{6}$.

$$\therefore\ S_1 = \frac{1}{1^2.2^2} + \frac{1}{2^2.3^2} + \frac{1}{3^2.4^2} + \ldots . \frac{\pi^2}{3} - 3.$$

$$S_2 = \frac{1}{1^2.2^2.3^2} + \frac{1}{2^2.3^2.4^2} + \frac{1}{3^2.4^2.5^2} + \ldots . \frac{\pi^2}{4} - \tfrac{39}{16}.$$

$$S_3 = \frac{1}{1^2.2^2.3^2.4^2} + \frac{1}{2^2.3^2.4^2.5^2} + \ldots . \frac{5\pi^2}{54} - \tfrac{197}{216}.$$

$$S_4 = \frac{1}{1^2.2^2.3^2.4^2.5^2} + \frac{1}{2^2.3^2.4^2.5^2.6^2} + \ldots . \frac{35\pi^2}{1728} - \tfrac{55225}{276648}.$$

$$S_5 = \frac{1}{1^2.2^2.3^2.4^2.5^2.6^2} + \frac{1}{2^2.3^2.4^2.5^2.6^2.7^2} + \ldots \frac{7\pi^2}{2400} - 3\tfrac{92211}{320000},$$

Etc. Etc. Etc. Etc.

[The above method is due to Mr. W. S. B. Woolhouse, F. R. A. S.]

S_4 is the value required by the problem.

II. Solution by the PROPOSER.

As $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} +$, etc., to infinity, equals $\pi^2/6$, we have

$$\frac{1}{1^2.2^2.3^2.4^2.5^2} + \frac{1}{2^2.3^2.4^2.5^2.6^2} + \frac{1}{3^2.4^2.5^2.6^2.7^2} + \ldots .$$

$$= \frac{5.6.7.8}{1^3.2^3.3^3.4^3}\left[\pi^2/6 - 3\left(\frac{1}{1.2} + \frac{1}{2.3.4} + \frac{1.2}{3.4.5.6} + \frac{1.2.3}{4.5.6.7.8}\right)\right]$$

$$= \tfrac{35}{288}(\pi^2/6 - \tfrac{1105}{672}) = .00007155904984233331 36+.$$

III. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

The series may be thrown into the form

$$\frac{1}{1^2}.\frac{1}{3^2}.\frac{1}{4^2}.\frac{1}{5^2}\left[\left(\frac{1}{1}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1.2}{6.7}\right)^2 + \left(\frac{1.2.3}{6.7.8}\right)^2 + \left(\frac{1.2.3.4}{6.7.8.9}\right)^2 + \ldots .\right]$$

$$= \frac{1}{2^2}.\frac{1}{3^2}.\frac{1}{4^2}.\frac{1}{5^2}\left[1 + \frac{1}{6^2} + \frac{1}{\left(\frac{6.7}{1.2}\right)^2} + \frac{1}{\left(\frac{6.7.8}{1.2.3}\right)^2} + \frac{1}{\left(\frac{6.7.8.9}{1.2.3.4}\right)^2} + \ldots .\right],$$

where the denominators are the squares of the successive figurate numbers of the 6th order. Whilst the sum of the reciprocals of the figurate numbers of any order may be expressed by a finite expression, those of the first two orders being excepted since their sums are infinite, the sum of the *squares* of the reciprocals of the figurate numbers cannot so be expressed with the exception of those of the second order, the latter being $\pi^2/6$. Since, however, the series of the giv-

en series converges rapidly, the sum may be found within any limits of precision. A few terms suffice to find the sum correctly within 7 or 8 decimals. The sum of the series without the squares would be exactly $\frac{1}{96}$.

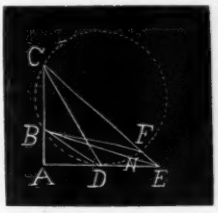
GEOMETRY.

94. Proposed by EDMOND FISH, Hillsboro, Ill.

A tower $AB=a$, is surmounted by a flag pole $BC=b$. A point D is so taken in a line perpendicular to the foot of the tower that angle BDC is a maximum. Prove that AD is a mean proportional between AC and AB.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, East Chester High School, Chester, Pa.; WALTER H. DRANE, A. M., Professor of Mathematics, Jefferson Military College, Washington, Miss.; C. A. JONES, Kosciusko, Miss.; E. R. GIBSON, Wayne, Neb.; and P. S. BERG, Larimore, N. D.

In order that $\angle BDC$ may be a maximum the line AD must be tangent to the circle through B, C at point D. If not let E be some other point in AD, and let EC cut the circle at F.



$\angle BFC > \angle BEC$. But $\angle BFC = \angle BDC$.

$\therefore \angle BDC > \angle BEC$. Q. E. D.

$\therefore AC : AD = AD : AB$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.; Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Me.; W. L. HARVEY, Portland, Me., and the PROPOSER.

Describing a circle passing through B and C and tangent to AD, the angle BDC is measured by one-half the arc BC, whereas for any other point E the angle BEC is measured by one-half the difference of the arcs BC and FN, therefore by a smaller arc than BC. And since AD is tangent and ABC a secant, we have $AD^2 = AC \times AB$.

By the Calculus we may prove this fact thus: Denoting AD by x, we have

$$\tan BDC = \tan(CDA - BDA) = \frac{\tan CDA - \tan BDA}{1 + \tan CDA . \tan BDA}$$

$$= \frac{\frac{a+b}{x} - \frac{a}{x}}{1 + \frac{a(a+b)}{x^2}} = \frac{bx}{x^2 + a(a+b)}.$$

By differentiating, this expression is a maximum for $x^2 = a(a+b)$.

III. Solution by EDWARD R. ROBINS, Master of Mathematics, Lawrenceville School, Lawrenceville, N. J.; and COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Take A as origin, $AB=a$, $BC=b$, $AD=c$.

Equation of BD is $x/c + y/a = 1$.

Equation of CD is $x/c + y/(a+b) = 1$.

$$\therefore \measuredangle BCD = \tan^{-1} \frac{-bc}{c^2 - a^2 - ab},$$

which will be a maximum when $c^2 - a^2 - ab = 0$, or $c^2 = a(a+b)$. Q. E. D.

95. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

At each point of a parabola is described the rectangular hyperbola of a four-pointic contact; prove that the locus of the center of the hyperbola is an equal parabola.

Solution by the PROPOSER.

The curve having four-pointic contact with the parabola $y^2 = 4ax$ .....(1)

is $y^2 - 4ax - \lambda(yy' - 2ax - 2ax')^2 = 0$ .....(2),

or, $-4\lambda a^2 x^2 + 4a\lambda y' xy + (1 - \lambda y'^2) y^2 - (4a + 8a^2 \lambda x') x + 4a\lambda x' y' y - 4a^2 \lambda x'^2 = 0$ ..(3).

If this be an equilateral hyperbola,

$$4\lambda a^2 = 1 - \lambda y'^2, \text{ or } \lambda = 1 \div (y'^2 + 4a^2) \text{ .....(4).}$$

Substituting this in (2) and reducing,

$$ax^2 - y'xy - ay^2 + (4a^2 + y'^2 + 2ax')x - x'y'y + ax'^2 = 0 \text{ ..... (5).}$$

The center of this is given by

$x = -(y'^2 + 8a^2)/4a$ ....(6), $y = y'$ ....(7). (x', y') being on (1), $y'^2 = 4ax'$ ....(8).

Eliminating x', y' from (6), (7), and (8), we have the required locus,

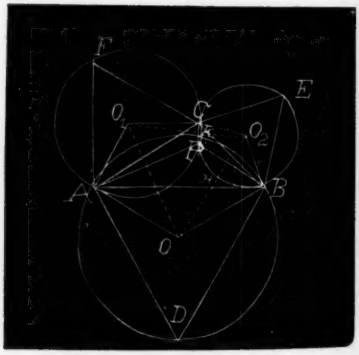
$$y^2 = -4a(x + 2a) \text{ .....(9).}$$

96. Proposed by W. F. BRADBURY, A. M., Head Master, Cambridge Latin School, Cambridge, Mass.

Isosceles triangles are constructed externally on the three sides of a triangle as bases, with the angles at the bases each 30°. The triangle formed by joining the remote vertices (the 120° vertices) of these isosceles triangles is equilateral. [Geometric—not Trigonometric—solution.]

Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

The vertices O, O_1, O_2, of the isosceles triangles are the centers of equilateral triangles described on the sides of ABC. Circumferences passed about these triangles intersect in a point, P.



Let P be the intersection of the two circles, AFC and CEB. Join AP, BP, and CP. Since $APCF$ is inscribed, $\angle F + \angle APC = 180°$. But $\angle F = 60°$.

$\therefore \angle APC = 120°$.

Similarly, $\angle CPB = 120°$.

$\therefore \angle APB = 120°$; and $\angle APB + \angle D = 180°$.

$\therefore APBD$ is inscribed, and P is in the circumference of ADB. Q. E. D.

Lines that join the centers of intersecting circles bisect the common

chords and the intercepted arcs. $\therefore$ arc $HP=\frac{1}{2}$ arc AP, and arc $PK=\frac{1}{2}$ arc PC. $\therefore$ arc $HPK=\frac{1}{2}$ arc APC. But arc APC measures $\angle F=60^\circ$ at circumference; therefore HK, its half, measures an equal angle at the center, and $\angle O_2=\angle F=60^\circ$. Similarly, $\angle O_1$ may be shown $=\angle E$, and $\angle O=\angle D$. Since equiangular triangles are equilateral, OO_1O_2 is equilateral. Q. E. D.

Solved in a similar manner by *G. I. HOPKINS, F. R. HONEY, J. SCHEFFER, C. A. JONES*, and *G. B. M. ZERR*.

CALCULUS.

73. Proposed by MOSES COBB STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Ind.

Solve $\int_0^{\frac{1}{4}\pi} \log(1+\tan x)dx$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

$$\int_0^{\frac{1}{4}\pi} \log(1+\tan x)dx=\int_0^{\frac{1}{4}\pi} \log[1+\tan(\tfrac{1}{4}\pi-x)]dx=\int_0^{\frac{1}{4}\pi} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$=\int_0^{\frac{1}{4}\pi} \log\left(\frac{2}{1+\tan x}\right)dx=\int_0^{\frac{1}{4}\pi}\log 2.dx-\int_0^{\frac{1}{4}\pi} \log(1+\tan x)dx.$$

$$\therefore 2\int_0^{\frac{1}{4}\pi} \log(1+\tan x)dx=\int_0^{\frac{1}{4}\pi} \log 2.dx=\tfrac{1}{4}\pi\log 2.$$

$$\therefore \int_0^{\frac{1}{4}\pi} \log(1+\tan x)dx=\tfrac{1}{8}\pi\log 2.$$

(See *Todhunter's Integral Calculus*, Art. 51, page 66.)

II. Solution by T. A. CLARK, of the Senior Class, Purdue University, Lafayette, Ind.

$$\int_0^{\frac{1}{4}\pi} \log(1+\tan x)dx=\int_0^{\frac{1}{4}\pi} \log(\tan\tfrac{1}{4}\pi+\tan x)dx=\int_0^{\frac{1}{4}\pi} \log\frac{\sin(\frac{1}{4}\pi+x)}{\cos\frac{1}{4}\pi\cos x}$$

$$=\int_0^{\frac{1}{4}\pi} \log\sec\tfrac{1}{4}\pi dx+\int_0^{\frac{1}{4}\pi}\sin(\tfrac{1}{4}\pi+x)dx-\int_0^{\frac{1}{4}\pi} \log\cos x dx=\tfrac{1}{8}\pi\log 2.$$

To prove $\int_0^{\frac{1}{4}\pi} \log\sin(\frac{1}{4}\pi+x)dx-\int_0^{\frac{1}{4}\pi} \log\cos x dx=0$. Put $\frac{1}{4}\pi+x=\theta$, and $\frac{1}{4}\pi-x=\phi$, then $\int_0^{\frac{1}{4}\pi} \log\sin(\frac{1}{4}\pi+x)dx=\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \log\sin\theta d\theta$, and $\int_0^{\frac{1}{4}\pi} \log\cos x dx$

$=-\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin\phi d\phi=\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin\phi d\phi$. Therefore it is evident $\int_0^{\frac{1}{2}\pi}\log\sin(\frac{1}{2}\pi+x)dx$

$-\int_0^{\frac{1}{2}\pi}\cos x dx=0$.

74. Proposed by EDWARD R. ROBBINS, A. B., Mathematical Master in the Lawrenceville School, Lawrenceville, N. J.

A circular ring, whose radii are a and b, is cut by a plane making the area of the section (or sections) a maximum. Required the position of the plane, and the nature and area of the section (or sections).

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

Let the axis of the surface be taken as the axis of z, then the equation to the ring is

$$(x^2+y^2+z^2+ab)^2=(a+b)^2(x^2+y^2).$$

The maximum section will be made by a plane passing through the center of the ring and making an angle β such that $\sin\beta<[(a-b)/(a+b)]$.

Let $z=x\tan\beta$ be this plane.

$\therefore (x^2\sec^2\beta+y^2+ab)^2=(a+b)^2(x^2+y^2)$.

The area of this section is $(\pi/\sec^2\beta)(a^2-b^2)$.

This is a maximum when $\sec^2\beta$ is least, $\therefore \beta=0$.

$\therefore (x^2+y^2+ab)^2=(a+b)^2(x^2+y^2)$.

$\therefore$ the plane coincides with the xy plane and divides the ring into two equal parts. The section is included between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$, area $=\pi(a^2-b^2)$.

MECHANICS.

64. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

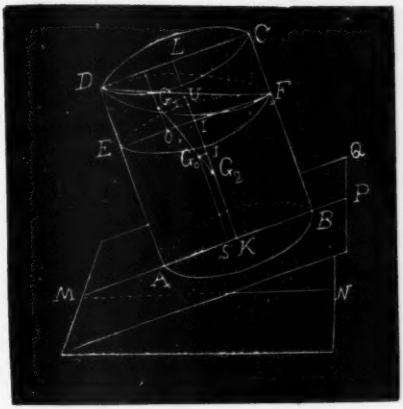
A cylindrical vessel, radius of vessel r and altitude h, is filled with water and rests on a horizontal plane. It is required to ascertain the maximum angle of elevation to which the plane may be raised without the vessel falling, allowing the coefficient of friction to be such as to prevent sliding, and the water to overflow as the plane is raised.

I. Solution by the PROPOSER.

Let MQ be the inclined plane; $ABCD$ a vertical section of the vessel; $ABFD$ a vertical section of the water when the vessel is on the point of turning over; $AB=2R$, the diameter of the vessel; $AD=H$, its altitude; and the angle $PMN=\theta$, maximum inclination of the plane. The quantity of water in the vessel at the time the vessel is on the point of turning over consists of the cylindrical part whose vertical section is $ABFE$ and the cylindrical ungula whose vertical section is EFD. Let V_2 be the volume of the cylindrical part, and V_1 the volume of the ungula. Let G_2 be the center of gravity of the cylindrical part, G_1 the center of gravity of the ungula, and G_0 the common center of gravity of both.

The vessel will be on the point of turning over when a vertical line thro' G_0 passes through A. Hence, the condition for the maximum inclination is

$$\frac{SA}{G_0S}=\tan\theta.$$



G_2 is on the axis of the cylinder and midway between K and I. Now $KI=KL-IL(=DE)=H-2R\tan\theta$. Hence $KG_2=\frac{1}{2}H-R\tan\theta$.

The center of gravity of the ungula is found as follows: Let FE be the positive x-axis, FC the positive z-axis, and a line perpendicular to these two lines at F the y-axis. Let $\bar{x}$, $\bar{y}$, and $\bar{z}$ be the coördinates of the center of gravity of the ungula. Since the plane EFD is a plane of symmetry, $\bar{y}=0$.

$$\bar{x}=\frac{\int_0^{2R}\int_0^{\sqrt{(2Rx-x^2)}}\int_0^{x\tan\theta}x\,dx\,dy\,dz}{\int_0^{2R}\int_0^{\sqrt{(2Rx-x^2)}}\int_0^{x\tan\theta}dx\,dy\,dz}=\frac{2}{\pi R^3\tan\theta}\int_0^{2R}\int_0^{\sqrt{(2Rx-x^2)}}\int_0^{x\tan\theta}x\,dx\,dy\,dz$$

$$=\frac{2}{\pi R^3}\int_0^{2R}\int_0^{\sqrt{(2Rx-x^2)}}x^2\,dx\,dy=\frac{2}{\pi R^3}\int_0^{2R}x^2(2Rx-x^2)^{\frac{1}{2}}\,dx,$$

$$=\frac{2}{\pi R^3}\left[\left(-\tfrac{1}{4}x(2Rx-x^2)^{\frac{3}{2}}\right)_0^{2R}+R\int_0^{2R}x\sqrt{(2Rx-x^2)}\,dx\right]$$

$$=\frac{2}{\pi R^2}\int_0^{2R}x\sqrt{(2Rx-x^2)}\,dx=\frac{2}{\pi R^2}\int_0^{2R}[R\sqrt{(2Rx-x^2)}-(R-x)\sqrt{(2Rx-x^2)}]\,dx$$

$$=\frac{2}{\pi R^2}\left[\frac{R(x-R)}{2}\sqrt{(2Rx-x^2}+\tfrac{1}{2}R^2\sin^{-1}\frac{x-R}{R}-\tfrac{1}{3}(2Rx-x^2)^{\frac{3}{2}}\right]_0^{2R}=\tfrac{5}{4}R=FO\ ;$$

$$\bar{z}=\frac{\int_0^{2R}\int_0^{\sqrt{(2Rx-x^2)}}\int_0^{x\tan\theta}z\,dx\,dy\,dz}{\int_0^{2R}\int_0^{\sqrt{(\frac{1}{2}Rx-x^2)}}\int_0^{x\tan\theta}dx\,dy\,dz}=\tfrac{5}{8}R\tan\theta=G_1O.$$

Hence, $G_1U=\frac{5}{4}R-R=\frac{1}{4}R$; $G_2U=G_2I+IU=\frac{1}{2}H-R\tan\theta+\frac{5}{8}R\tan\theta$

$=\frac{1}{8}(4H-3R\tan\theta)$; and $G_1G_2=\frac{1}{8}\sqrt{(G_1U^2+G_2U^2)}=\frac{1}{8}\sqrt{[(4H-3R\tan\theta)^2+4R^2]}$.

Taking moments about G_0, we have $G_1G_0\times V_1=G_0G_2\times V_2$, the masses being proportional to the volumes. $\therefore G_1G_0 : G_0G_2=V_2 : V_1$, or $G_1G_2 : G_0G_2 =V_2+V_1 : V_1$, or $\frac{1}{8}\sqrt{[4H-3R\tan\theta)^2+4R^2]} : G_0G_2=\pi R^2(H-R\tan\theta) : \pi R^3\tan\theta$.

$\therefore G_0G_2=R\tan\theta\sqrt{[(4H-3R\tan\theta)^2+4R^2]}/8(H-R\tan\theta)$.

$G_1U : G_1G_2=G_0J(=SK) : G_0G_1$, whence $G_0J=SK=R^2\tan\theta/4(H-R\tan\theta)$.

$$\therefore SA=R-\frac{R^2\tan\theta}{4(H-R\tan\theta)}=\frac{R(4H-5R\tan\theta)}{4(H-R\tan\theta)}.$$

$G_1U : UG_2=G_0J : JG_2$, whence $JG_2=\dfrac{R\tan\theta(4H-3R\tan\theta)}{8(H-R\tan\theta)}$.

$$\therefore JK=G_0S=\frac{R(4H-3R\tan\theta)\tan\theta}{8(H-R\tan\theta)}+(\tfrac{1}{2}H-R\tan\theta)$$

$$=\frac{R(4H-3R\tan\theta)\tan\theta+8(\frac{1}{2}H-R\tan\theta)(H-R\tan\theta)}{8(H-R\tan\theta)}.$$

$$\frac{SA}{G_0S}=\tan\theta=\frac{R(4H-5R\tan\theta)}{4(H-R\tan\theta)}\Big/\frac{R(4H-3R\tan\theta)\tan\theta+8(\frac{1}{2}H-R\tan\theta)(H-R\tan\theta)}{8(H-R\tan\theta)}$$

$$\text{or } \tan\theta=\frac{2R(4H-5R\tan\theta)}{R(4H-3R\tan\theta)\tan\theta+(4H-8R\tan\theta)(H-R\tan\theta)},$$

whence $5R^2\tan^3\theta-8RH\tan^2\theta+2(2H^2+R^2)\tan\theta-8RH=0$, an equation from which $\tan\theta$ can be found, and, therefore, θ.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let the required maximum angle be denoted by θ. $ABCD$ represents the vessel, $ABFD$ the quantity of water, and G_0 the center of gravity of the portion $ABFD$. Let $SA=x$ be any portion of the diameter AB, and x_0 and z_0 be the coordinates of the center of gravity G_0, in the direction of AB and AC, and V the volume of $ABFD$. Then, since $CF=2r\tan\theta$, and SQ' (Q' does not appear in the above figure. It is the intersection of SG_0 and DF) $=h-x\tan\theta$,

$$V_{x_0}=2\int_0^{2x}x(h-x\tan\theta)\sqrt{(2rx-x^2)}\,dx=\frac{\pi r^3}{8}(4h-5hx\tan\theta).$$

$$V_{z_0}=\int_0^{2r}(h-x\tan\theta)\sqrt{(2rx-x^2)}\,dx=\frac{\pi r^2}{8}(8h^2-8hr\tan\theta+5r^2\tan^2\theta).$$

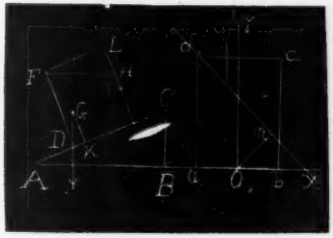
The limiting position of equilibrium is that for which the vertical from G_0 will meet A, or $\tan\theta=x_0/z_0$.

$\therefore$ We obtain the final equation

$$5r^2\tan^3\theta - 8hr\tan^2\theta + 2(2h^2+5r^2)\tan\theta - 8hr = 0.$$

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

First find the center of gravity of the portions of the cylinder between ab and dm. Let $\angle cdm = \beta$, $ab = 2r$, $bc = h$. Then equation to the cylinder is $x^2+z^2 = r^2$; of line dm, $x\sin\beta + y\cos\beta = h\cos\beta - r\sin\beta$.



Let (l, k) be the coördinates of the center of gravity.

The limits of x are $-r$ and $+r$; of y, 0 and $[h-(r+x)\tan\beta] = y'$; of z, 0 and $\sqrt{(r^2-x^2)} = z'$.

$$\therefore\ l = \frac{\int_{-r}^{r}\int_{0}^{y'}\int_{0}^{z'} x\,dx\,dy\,dz}{\int_{-r}^{r}\int_{0}^{y'}\int_{0}^{z'} dx\,dy\,dz} = \int_{-r}^{r} xy'z'\,dx \Big/ \int_{-r}^{r} y'z'\,dx$$

$$k = \frac{\int_{-r}^{r}\int_{0}^{y'}\int_{0}^{z'} y\,dx\,dy\,dz}{\int_{-r}^{r}\int_{0}^{y'}\int_{0}^{z'} dx\,dy\,dz} = \tfrac{1}{2}\int_{-r}^{r} y'^2z'\,dx \Big/ \int_{-r}^{r} y'z'\,dx.$$

$$\therefore\ l = -\frac{r^2\tan\beta}{4(h-r\tan\beta)},\qquad \frac{4h^2-8hr\tan\beta+5r^2\tan^2\beta}{8(h-r\tan\beta)} = k.$$

Now $LFH = DGK = CAB = \beta$.

$$\therefore\ \tan\beta = DK/GK = \frac{r-l}{k} = \frac{8hr-10r^2\tan\beta}{4h^2-8hr\tan\beta+5r^2\tan^2\beta}.$$

Let $u = \tan\beta$.

$$\therefore\ u^3 - \frac{8h}{5r}u^2 + \frac{4h^2+10r^2}{5r^2}u = \frac{8h}{5r}.$$

Let $u = v + (8h/15r)$.

$$\therefore\ v^3 + \frac{150r^2-4h^2}{75r^2}v + \frac{3246h^3-1800hr^2}{3375r^3} = 0.$$

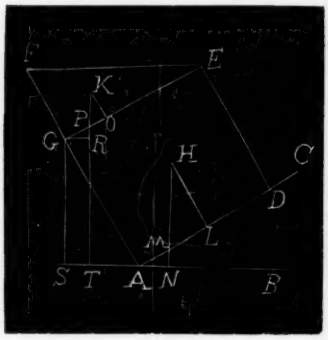
Let $\frac{150r^2-4h^2}{75r^2}=3m$, and $\frac{3246h^3-1800hr^2}{3375r^3}=2n$.

$\therefore\ v^3+3mv+2n=0.\quad v=[-n+\sqrt{(n^2+m^3)}]^{\frac{1}{3}}+[-n-\sqrt{(n^2+m^3)}]^{\frac{1}{3}}$.

In the above, the weight of the vessel is disregarded.

IV. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Miss.

Let AC be the inclined plane making an angle β with the horizontal AB, $ADEF$ the water in the vessel, EG parallel to AD, H and K the centers of gravity of the parts $ADEG$ and EGF respectively. The construction of the rest of the figure is evident.



Denote AF, AG, and AD by h, $2a$, and $2r$, respectively. Neglecting the weight of the vessel, it will be on the point of overturning when $ADEG\times AN=EFG\times AT$.....(1).

To find the center of gravity of EFG, consider EG the positive direction of the x-axis, ED the negative direction of the z-axis, and a perpendicular to these the y-axis. The solid question is bounded by the cylindrical surface $y^2=2rx-x^2$, and the planes $z=0$, $z=\tan\beta.x$. Since the xz-plane is one of symmetry, $\bar{y}=0$.

$$\bar{x}=\frac{\int_0^{2r}\int_0^{y_1}\int_0^{z_1}x\,dx\,dy\,dz}{\int_0^{2r}\int_0^{y_1}\int_0^{z_1}dx\,dy\,dz},\qquad \bar{z}=\frac{\int_0^{2r}\int_0^{y_1}\int_0^{z_1}z\,dx\,dy\,dz}{\int_0^{2r}\int_0^{y_1}\int_0^{z_1}dx\,dy\,dz},$$

where $z_1=\tan\beta.x$, $y_1=\sqrt{(2rx-x^2)}$.

From these, $\bar{x}=\frac{3}{4}r=EO$, $\bar{z}=\frac{4}{3}r\tan\beta=OK$.

$OP=OK\tan\beta=\frac{4}{3}r\tan^2\beta$;

$GP=GE-OE-OP=\frac{3}{4}r-\frac{4}{3}r\tan^2\beta$;

$GR=GP\cos\beta=\frac{3}{4}r\cos\beta-\frac{4}{3}r\cos\beta\tan^2\beta$;

$SA=AG\sin\beta=2a\sin\beta$;

$AT=SA-GR=2a\sin\beta-\frac{3}{4}r\cos\beta+\frac{4}{3}r\cos\beta\tan^2\beta$.

$ML=HL\tan\beta=a\tan\beta$;

$AM=AL-ML=r-a\tan\beta$;

$AN=AM\cos\beta=r\cos\beta-a\sin\beta.$

Volume of $ADEG=2\pi ar^2.$

$$\text{Volume of } EGF=2\int_0^{2r}\int_0^{\sqrt{(2rx-x^2)}}\int_0^{x\tan\beta}dxdydz=\pi r^3\tan\beta.$$

[This result comes easily without the calculus].

Substituting in equation (1), transposing, dividing through by $\cos\beta$, rearranging, and clearing of fractions,

$$5r^2\tan^3\beta+16ar\tan^2\beta+2(8a^2-3r^2)\tan\beta-16ar=0.$$

Substituting in this, $\frac{1}{2}h-r\tan\beta$ for a, and reducing,

$$5r^2\tan^3\beta-8rh\tan^2\beta+2(5r^2+2h^2)\tan\beta-8rh=0,$$

from which β may be found.

AVERAGE AND PROBABILITY.

62. Proposed by O. S. KIBLER, Superintendent of Schools, Middleburg, O.

A bag contains any number of balls, which are equally likely to be white or black; one is drawn and found to be white. Show that the chance of drawing another white one, the first ball not being replaced, is two-thirds. [From *C. Smith's Treatise on Algebra*, page 615].

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

If m balls are drawn and turn out white, the chance that n others drawn will be white is: $p=[(m+1)/(m+n+1)]$. In the problem, $m=n=1$.

$\therefore p=\frac{2}{3}$. Otherwise, $p=\int_0^1 x^2dx/\int_0^1 xdx=\frac{2}{3}$.

This is the simplest case of the article on page 107, No. 4, Vol. II. of the MONTHLY.

64. Proposed by REV. W. A. WHITWORTH, A. M.

O is a given point within a triangle; P is a random point within the same. The line through O and P is produced so as to divide the triangle into a trapezium and a triangle. Find the average area of this triangle. [From the *Educational Times*, London, Eng.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

Let $\triangle$ be the average area required, $\triangle_1$ the average area of BFG; (m, n) coördinates of O. Then the coördinates of E are $(b, bn/m)$; of D, $[b^2n/(ab-am+bn), abn/(ab-am+bn)]$.

Area $ABC=\frac{1}{2}ab\sin C$.

Area $ACO=\frac{1}{2}bn\sin C$.

Area $ADO=bn(am-bn)\sin C/[2(ab-am+bn)]$.

Area $COE=bn(b-m)\sin C/(2m)$.

Area $DOEB=ABC-(ACO+ADO+COE)$.

$\therefore$ Area $DOEB=(a^2b^2m-a^2bm^2+2ab^2mn-abm^2n+b^2n^2m$
$-ab^3n-b^3n^2)\sin C/[2m(ab-am+bn)]$.

The area $DOEB$ does not vary.

The average area of $GOE+DOF=\frac{1}{2}(COE+AOD)$

$$=\frac{(2abm^2n-2b^2mn^2-2ab^2mn+ab^3n+b^3n^2)\sin C}{4m(ab-am+bn)}.$$

$\therefore\ \triangle_1=DOEB+\frac{1}{2}COE+AOD.$

$$\therefore\ \triangle_1=\frac{(2a^2b^2m-2a^2bm^2+2ab^2mn-ab^3n-b^3n^2)\sin C}{4m(ab-am+bn)}.$$

Similarly, $\triangle_2=\dfrac{(2b^2c^2m-2b^2cm^2+2bc^2mn-bc^3n-c^3n^2)\sin C}{4m(cb-bm+cn)}$.

$$\triangle_3=\frac{(2a^2c^2m-2ac^2m^2+2a^2cmn-a^3cn-a^3n^2)\sin C}{4m(ac-cm+an)}.$$

$\triangle=\frac{1}{3}(\triangle_1+\triangle_2+\triangle_3)$. If $a=b=c$ and $m=2n=\frac{2}{3}a$, $\triangle=\frac{1}{3}(a^2\sqrt{3})=\frac{1}{2}$ area of the triangle.

DIOPHANTINE ANALYSIS.

65. Proposed by MANSFIELD MERRIMAN, C. E., Ph. D., Professor of Civil Engineering, Lehigh University, South Bethlehem, Pa.

Show that the number 1521 can be expressed in seven different ways as the sum of three perfect squares. Can more than seven different ways be found?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

With the aid of a table that I devised for finding the sum of two squares equal to a number, I found 1521 expressed in *eight* different ways as the sum of three perfect squares. $1521=39^2=2^2+19^2+34^2=2^2+26^2+29^2=9^2+12^2+36^2=10^2+14^2+35^2=13^2+14^2+34^2=13^2+26^2+26^2=14^2+22^2+29^2=19^2+22^2+26^2$.

II. Solution by CHARLES CARROLL CROSS, Libertytown, Md.

The terminal figures of square numbers are, 0, 1, 4, 5, 6 and 9. These may be combined,—taking three at a time, so that the terminal figure will be one in the following manner: (5+5+1)..... (1); (9+1+1).....(2); (1+0+0)..... (3); (9+6+6).....(4); (6+4+1)..... (5); and (6+5+0)..... (6). It readily appears that it is impossible to combine any square numbers represented by (1), (2) and (3) so as to make 1521.

Those of (4) are found to be $13^2+34^2+14^2$ and $13^2+26^2+26^2$; of (5), $34^2+2^2+19^2$, $26^2+22^2+19^2$, $26^2+2^2+29^2$ and $14^2+22^2+29^2$; and of (6), $14^2+36^2+10^2$,—seven in all.

$\therefore\ 1521=13^2+34^2+14^2=13^2+26^2+26^2=34^2+2^2+19^2=26^2+22^2+19^2$
$=26^2+2^2+29^2=14^2+22^2+29^2=14^2+35^2+10^2$.

III. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Let $1521=m^2$, and x^2, y^2, and z^2 represent the three squares; then $x^2+y^2+z^2=m^2$, and $x^2=m^2-(y^2+z^2)=m^2-2pm+p^2$. $2pm=p^2+y^2+z^2$. Let $y=tp$ and $z=sp$, then $2m=p(s^2+t^2+1)$. Restoring the value of m, $p^2=78/(s^2+t^2+1)$, in which s and t may be any rational numbers. Take $s=1$, $t=1$, then $p=26$; $x=m-1=13$, $y=tp=26$ and $z=sp=26$, and $13^2+26^2+26^2=1521$. Take $s=2$, and $t=1$, $p=13$; $x=26$, $y=13$, $z=26$. Take $s=3$, $t=4$, then $p=3$; $x=36$, $y=12$, $z=9$. Take $s=2$, $t=3$, $p=\frac{8}{7}\frac{9}{}$; $x=\frac{2}{7}\frac{3}{}\frac{4}{}$, $y=\frac{7}{7}\frac{8}{}$, $z=\frac{1}{7}\frac{1}{}\frac{7}{}$, and $[(234)^2+(78)^2+(117)^2]/49=1521$.

While this is a solution of the question read *literally*, of course, I understand that the proposer intends to call for integral numbers; but I have obtained seven integral results, only *by trial*.

Also solved by *SYLVESTER ROBINS*, and *G. B. M. ZERR*.

66. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find two cubic proper fractions whose product is a square proper fraction. Can a *general* solution be made?

Solution by E. L. SHERWOOD, A. M., Superintendent City Schools, West Point, Miss.; CHARLES CARROLL CROSS, Libertytown, Md.; and G. B. M. ZERR, A. M., Ph. D., Chester High School, Chester, Pa.

Let a^6/b^6, c^6/d^6 be the fractions, $a<b$, $c<d$.
Then $a^6c^6/b^6d^6=(a^3c^3/b^3d^3)^2$.
Let $a=1$, $b=2$, $c=2$, $d=3$.
$\therefore\ a^6/b^6=\frac{1}{6}\frac{}{4}$, $c^6/d^6=\frac{6}{7}\frac{4}{2}\frac{}{9}$;
$\therefore\ a^6/b^6=(\frac{1}{4})^3$, $c^6/d^6=(\frac{4}{9})^3$; $a^6c^6/b^6d^6=(\frac{1}{2}\frac{}{7})^2$.
Other fractions can easily be found.

Also solved by *J. H. DRUMMOND*.

67. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Find (1) four consecutive numbers whose sum is a square, and (2) four consecutive numbers the sum of whose squares is a square.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let $x-1$, x, $x+1$ and $x+2$=four consecutive *integers*.

(1) Then their sum$=4x+2=2(2x+1)=2$ *times an odd number*. But this result can never be a square. $\therefore$ The sum of four consecutive integers can *not* be a square.

(2) The sum of their squares$=4x^2+4x+6=2[2(x^2+x+1)+1]=2$ *times an odd number*, which result can never be a square. $\therefore$ The sum of the squares of four consecutive integers can *not* be a square.

If, however, four consecutive numbers may be considered as four *fractions* whose denominators are the same number and equal to 2 *times a square*, and

whose numerators are consecutive integers, then we are able to fulfill the *first part* of the problem.

Of any four consecutive integers we have shown that their sum is 2 *times an odd number*. Now when this odd number is a *square*, we can find four consecutive fractions whose sum is a square, by making the denominators$=2m^2$ and the respective numerators$=2n(n+1)-1$, $2n(n+1)$, $2n(n+1)+1$, and $2n(n+1)+2$. Whence we have $\{[2n(n+1)-1]/2m^2\}+\{[2n(n+1)]/2m^2\}+\{[2n(n+1)+1]/2m^2\}+\{[2n(n+1)+2]/2m^2\}=(2n+1)^2/m^2$.

When $n=m=1$, we find $\frac{3}{2}+\frac{4}{2}+\frac{5}{2}+\frac{6}{2}=3^2$. When $n=1$ and $m=2$, we have $\frac{3}{8}+\frac{4}{8}+\frac{5}{8}+\frac{6}{8}=(\frac{3}{2})^2$. When $n=2$ and $m=1$, we find $\frac{11}{2}+\frac{12}{2}+\frac{13}{2}+\frac{14}{2}=5^2$, etc.

II. Solution by CHARLES CARROLL CROSS, Libertytown, Md.

(1) Combining the consecutive numbers we find that $1+2+3+4$, $5+6+7+8$, $6+7+8+9$, $7+8+9+0$, and $0+1+2+3$ are the only combinations whose terminal figure produces the terminal figure of a square. The first and third combinations can never produce a square number, because a square number whose terminal figure is 0 is always preceded by 0. The second and last combinations cannot produce a square number, because a square number whose terminal figure is 6 is always preceded by an odd number. The fourth combination can never be a square number, because a square number whose terminal figure is 4 is always preceded by an even number. Hence (1) is incorrect.

(2) In the *Mathematical Visitor*, Vol. I, No. 5, page 156, Dr. Martin has shown that the sum of three, of four, and of five consecutive squares, cannot be a square number. Hence (2) is also incorrect. [See also *Mathematical Magazine*, Vol. II, No. 6, page 92.]

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

(1) Let x, $x+1$, $x+2$, $x+3$ be the numbers.

$\therefore 4x+6=a^2$, or $x=(a^2-6)/4$.

$\therefore (a^2-6)/4$, $(a^2-2)/4$, $(a^2+2)/4$, $(a^2+6)/4$ are the numbers.

(2) $4x^2+12x=b^2-14$.

$\therefore x=\frac{1}{2}\sqrt{(a^2-5)}-\frac{3}{2}$, where (a^2-5) must be a square.

Let $a=3$, then $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, are the numbers.

Let $a=2\frac{1}{4}$, then $-\frac{7}{6}$, $-\frac{1}{6}$, $\frac{5}{6}$, $\frac{11}{6}$ are the numbers.

And so for other values of a.

Also solved by *EDWARD R. ROBBINS*, and *J. H. DRUMMOND*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

100. Proposed by CHARLES C. CROSS, Libertytown, Md.

I bought stock at 4% discount, and sold it at 2½% premium, paying a brokerage in both cases of ¼%. If my net profits were $130, what was my investment? (Solve by Arithmetic.)

101. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A man gained m=3% on his money, in July; and, in August, lost n=2%. What per cent. of his money July 1st is his money September 1st?

*** Solutions of these problems should be sent to B. F. Finkel, not later than November 10.

ALGEBRA.

89. Proposed by G. A. MILLER, Ph. D., Instructor in Mathematics, Cornell University, Ithaca, N. Y.

Solve by quadratics.,

$$x^2+y=7.....(1).$$
$$x+y^2=11.....(2).$$

*** Solutions of this problem should be sent to J. M. Colaw, not later than November 10.

GEOMETRY.

102. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Ohne Benutzung des Cirkels eine Strecke AC zu halbiren, wenn eine Parallele der Geraden AC gegeben ist. [*Reye's Geometrie der Lage*, Part I, p. 191.]

103. Proposed by FREMONT CRANE, Sand Coulee, Mont.

A horse is tethered with a rope which is attached to a stake B on the edge of a circular pond containing one acre. How long must the rope be to allow the horse to graze over one acre? [From *Home Study Magazine*, problem 249.]

104. Proposed by SAMUEL E. HARWOOD, M. A., Professor of Mathematics, Southern Illinois State Normal University, Carbondale, Ill.

To find a point in the circumference of a semi-circumference such that the sum of its distances from the extremites of the diameter shall be a maximum. [From *Wentworth's Plane Geometry*, Ex. 387.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than November 10.

CALCULUS.

80. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A vessel is anchored in three fathoms of water, and the cable passes over a sheave in

the bowsprit, which is six feet above the water. If the cable is hauled in at the rate of a foot a second, how fast is the vessel moving through the water when there is five fathoms of cable out? What is the acceleration of the vessel's velocity? [From *Byerly's Problems in Differential Calculus.*] Ans.—(*a*) 5/6 feet per second; (*b*)12/121 feet per second. Are these results correct?

*** Solutions of this problem should be sent to J. M. Colaw, not later than November 10.

MECHANICS.

73. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax Schools, Pittsburg, Pa.

A sixteen-foot plank weighs thirty-two pounds and is supported by two props, four feet and two feet from the ends. What weight is supported by each prop?

74. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

In the experiment of swinging in a vertical circle a glass containing water, and suspended by means of a string, if the string be two feet long, what must be the velocity at the lowest point if the experiment is to succeed? [From *Ziwet's Theoretical Mechanics*, Part III., p. 96.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than November 10.

AVERAGE AND PROBABILITY.

69. Proposed by Rev. W. ALLEN WHITWORTH, M. A.

There are n equal sugar sticks. Each stick is broken into two pieces, all positions of the fracture being equally likely. Of the two n pieces thus formed, a child is to take the largest. Show that his expectation is $[2n+1]/[2(n+1)]$ of a stick. [From *The Educational Times*, June, 1898.]

70. Proposed by Professor MILLER.

A ship at A observes another at B, whose course is unknown. Supposing their speed the same, prove that the chance of their coming within a given distance, d, of each other is always $(2/\pi)\sin^{-1}(d/a)$, whatever the course taken by A; provided its inclination to AB is not greater than $\cos^{-1}(d/a)$, where $AB=a$.

[From *Cambridge Mathematical Tripos*, 1871.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than November 10.

EDITORIALS.

F. W. Hanawalt, formerly of DePauw University, Greencastle, Ind., has been elected Professor of Mathematics in Iowa Wesleyan University, Mount Pleasant, Iowa.

E. M. Shepard, A. M., Professor of Biology in Drury College, is off on leave of absence, visiting the Sandwich Islands, the Philippines, New Guinea, Australia, and Samoa, in the interest of his department.

J. H. Tanner, of Cornell University, and Joseph Allen, of the College of the City of New York, have prepared an Analytical Geometry, which, judging from advanced sheets sent us through the courtesy of the publishers, the American Book Company, promises to be a very excellent work.

Our valued contributor, E. D. Roe, Jr., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio, had conferred upon him in July the degree of Doctor of Philosophy, *magna cum laude*, by the University of Erlangen, Germany; the subject of his thesis being, *Entwickelung der Sylvester'schen Determinante nach Normal-formen.*

BOOKS AND PERIODICALS.

Elements of Plane and Spherical Trigonometry. By J. W. Nicholson, A. M., LL. D., Professor of Mathematics, Louisiana State University and Agricultural and Mechanical College. 8vo., cloth. Pp. 101+pp. 62 of tables. Price, $1.10. New York and Chicago: The Macmillan Co.

This work presents only the common and most essential elements, it not being the author's aim to write a treatise. It comprises, therefore, what is needed to meet the wants of our best colleges and schools. Among the new features of the book are to be found: the *Trigonometric Circle*, and the introduction of the terms *opposite*, *adjacent* and *like functions*. The mechanical execution of the work bears the "stamp" of the publishers.

B. F. F.

Introduction to Algebra for the Use of Secondary Schools and Technical Colleges. By G. Chrystal, M. A., LL. D., Honorary Fellow of Corpus Christi College, Cambridge; Professor of Mathematics in the University of Edinburgh; Author of "An Elementary Text-Book on Algebra," Two Vols. 8vo. cloth. xxv+412 pp. London: Adam & Chas. Black.

Those who are acquainted with Chrystal's large work on algebra will be interested to examine the present work. Any short notice of it is inadequate to give a clear notion of its merits. There is one criticism that may be offered, viz.: the printed matter is too closely packed, thus marring the appearance of the printed page. B. F. F.

Infinitesimal Analysis. By William Benjamin Smith [Ph. D., Görtingen]. Vol. I. Elementary: Real Variables. Large 8vo. cloth. 352 pages. Price, $3.25. New York: The Macmillan Co.

This volume is in line of excellence with the other works written by Dr. Smith. His *Introductory Modern Geometry of the Point, Ray and Circle,* and his *Co-ordinate Geometry* are books of the highest order, both being written in the light and spirit of modern mathematical teaching. The *Infinitesimal Analysis* treats somewhat in detail the elementary theory and application of the calculus, laying its foundation strong by rigorous argument. The author has given more than usual attention to Hyperbolic Functions, Maxima and Minima, Operators, Tortuous Curves, Partial Derivatives, Multiple Integration, Jacobians, Gamma Functions, etc. The work is one of great interest and value, and will add additional weight to American authorship in this fascinating field of inquiry. It is to be hoped that the great demand for this volume will be an inspiration to its author, and encourage him in the ardious labor of writing Volume II, which, we trust, will soon follow.

B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hundson, N. Y.

A first-class magazine, so low in price as to make it possible to be in every home.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.00 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

The *American Monthly Review of Reviews* for October gives special attention to the developments of the past month in international politics and to the lessons of the Spanish-American war. The editor, in the department of "The Progress of the World," discusses the attitude of the Spanish people toward peace conditions, the new relations between Germany and England, the Czar's proposition for disarmament, the Dreyfus case in France, England's reopening of the Soudan. and other serious problems confronting the European powers. Important contributed articles review President McKinley's course in the conduct of the war to a successful close and the deficiencies in our administrative machinery revealed by the fatal delays and break-downs in the medical and subsistence departments of army management.

ERATTA.

June—July Number.

Page 164, line 19, for "$\sqrt{1-n^2}$" read $\sqrt{1+n^2}$.
Page 165, line 9, for "(" read).
Page 168, line 2, for "$(a+cm)^2$" read $(a+cm)^3$.
Page 168, line 12, for "$(c-y)$" read $)c--y)^2$.
Page 168, line 17, for "$\frac{1}{9}\pi$" read $\frac{1}{2}\pi$.
Page 169, line 10, omit].
Page 169, line 12, insert] at end of line.
Page 169, line 15, for "am^2" read a^2m.
Page 169, line 16, for "$(R+r)$" in denominator, read $(R-r)$.
Page 169, line 18, insert] at end of line.
Page 170, line 4 insert R^2 before $\sin^{-1}$.